16/03/2022

SET THEORY

Set – Well defined collection of elements.

Representation –

1. Roster – $A = \{1, 2, 3, 4, 5\}$
2. Set Builder – $A = \{x : \text{Some condition on } 'x'\}$

Standard Sets –

$\mathbb{N} = \{1, 2, \ldots\}$

$\mathbb{Z} / I = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

$\mathbb{Q} = \{x : x = \frac{p}{q}, q \neq 0, p, q \in \mathbb{Z}\}$

$\mathbb{R} = \{x : x \in \mathbb{R}\}$

Imp. sets —

(+ve) Integers ($\mathbb{Z}^+$) = $\mathbb{N}$

(−ve) Integers ($\mathbb{Z}^-$) = $\{\ldots, -2, -1\}$

(+ve) Real nos. ($\mathbb{R}^+$) = $\{x : (x > 0) \wedge (x \in \mathbb{R})\}$

(−ve) Real nos. ($\mathbb{R}^-$) = $\{x : (x < 0) \wedge (x \in \mathbb{R})\}$

Non (+ve) Real nos. = $\mathbb{R}^- \cup \{0\}$

Non (−ve) Real nos. = $\mathbb{R}^+ \cup \{0\}$

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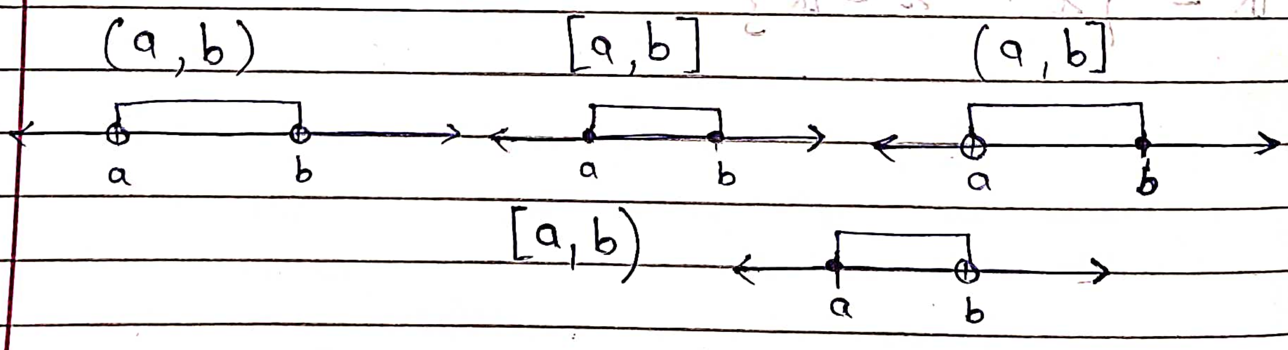
Interval —

Open (a, b)

Closed $[a, b]$

Semi Open/ Semi Close $[a, b)$ $(a, b]$



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Inequality —

Less than $<$ Less than or equal to $\leq$

Greater than $>$ Greater than or equal to $\geq$

Types of Sets —

1. Finite : Empty or Definite no. of elements.
2. Infinite : Infinite no. of elements.
3. Empty : No element. Symbol is $\emptyset$ or $\{\}$.
4. Singleton : Single element.
5. Equivalent : $n(A) = n(B)$, Eg: $A = \{1, 2\}$; $B = \{a, b\}$
6. Equal : All elements are same.

Subset —

If every element of A is an element of B, then A is called subset of B.

A subset of B $\Rightarrow$ $A \subseteq B$

☆ $\boxed{(A \subseteq B) \wedge (B \subseteq A) \Leftrightarrow (A = B)}$

☆ Proper Subset : $(A \subseteq B) \wedge (A \neq B) \Leftrightarrow (A \subset B)$

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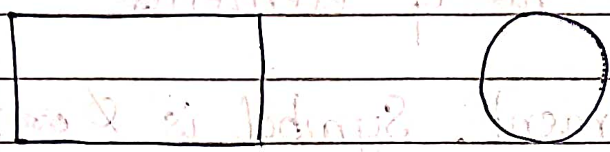
Universal Set —

If there are some sets under consideration, then there happens to be a set which is superset of each of the given sets. Such a set is called universal set. Symbol 'U'.

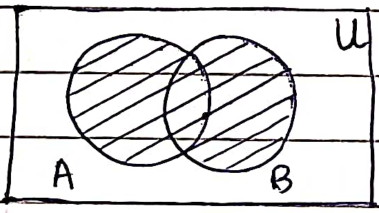
Venn Diagram —



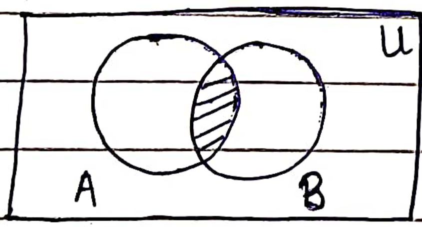
Universal Set   Set

Operations on Sets —

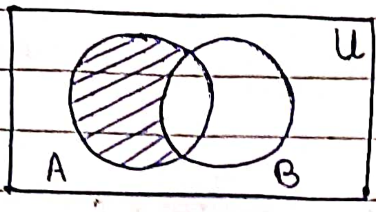
1. Union [U] : $\boxed{A \cup B = \{x : (x \in A) \vee (x \in B)\}}$



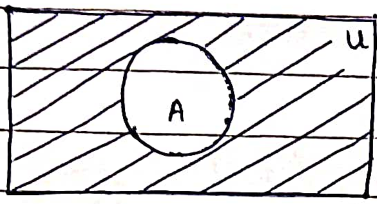
2. Intersection [∩] : $\boxed{A \cap B = \{x : (x \in A) \wedge (x \in B)\}}$



3. Difference [$-$] : $\boxed{A-B=\{x:(x\in A)\wedge(x\notin B)\}}$



4. Complement [$\bar{\ }$] : $\boxed{\bar{A}=(U-A)}$



Prop^{ts} –

i) $\overline{(\bar{A})}=A$

ii) $A\cup\bar{A}=U$

iii) $A\cap\bar{A}=\phi$

iv) $\bar{U}=\phi$

v) $\bar{\phi}=U$

vi) $\boxed{\overline{(A\cap B)}=\bar{A}\cup\bar{B}}$

vii) $\boxed{\overline{(A\cup B)}=\bar{A}\cap\bar{B}}$

→ De Morgan's Laws

Principle of Inclusion & Exclusion –

$$\boxed{n(A\cup B)=n(A)+n(B)-n(A\cap B)}$$

$$\boxed{n(A\cup B\cup C)=\begin{pmatrix}n(A)+n(B)+n(C)\\-n(A\cap B)-n(B\cap C)-n(C\cap A)\\+n(A\cap B\cap C)\end{pmatrix}}$$

$$\boxed{n(A_1\cup\ldots A_n)=\sum(n(A_i))-\sum(n(A_i\cap A_j))+\ldots}$$

Modulus

Def^n: 1.) $|x|$; Dist. of real no. 'x' from origin on no. line.

2.) $|x| = \sqrt{x^2}$

3.) $|x| = \max\{x, -x\}$

4.) $|x| = \begin{cases} x \; ; & \text{if } x \geq 0 \\ -x \; ; & \text{if } x < 0 \end{cases}$

Prop^{t}s –

1) $||x|| = |x|$

2) $|x| = a$, $a \in \mathbb{R}$
if $a < 0$; $x \in \phi$ (No sol^n)
if $a \geq 0$; $x \in \{a, -a\}$

3) $|x| = |y|$
$\Rightarrow$ $x = \pm y$

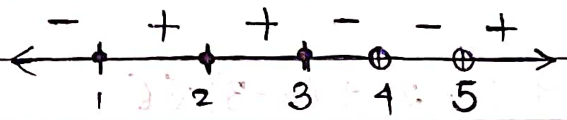
4) $|x| \leq a$; $a \in \mathbb{R}^+$
$\Rightarrow$ $x \in [-a, a]$

5) $|x| \geq a$; $a \in \mathbb{R}^+$
$\Rightarrow$ $x \in (-\infty, -a] \cup [a, \infty)$

Wavy Curve Method –

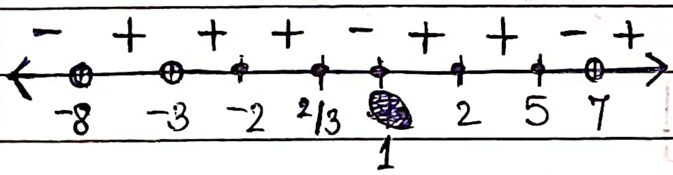
1) Factor Num^r & Denom^r seperately.
2) Calculate the ZEROS of Num^r & Denum^r seperately.
3) Plot all the zeroes on a no. line in increasing order.
4) If we have zeroes with even multiplicity, they DON'T change the sign of wave.
5) If we have zeroes with odd multiplicity, they DO change the sign of wave.
6) Zeroes of denum^r are NEVER included in the sol^n.

Q) $\dfrac{(x-1)(x-2)^2(x-3)^3}{(x-4)^4(x-5)^3} \leq 0$.

A) 

$x \in (-\infty, +1] \cup [3,4) \cup (4,5) \cup \{2\}$

Q) $\dfrac{(x-1)(x^2+x+1)(x+2)^2(3x-2)(x-5)^9|x-2|}{(x+3)^2(x-7)^9(x+8)^{11}} \leq 0$

A) 

$x \in (-\infty, -8) \cup [2/3, 1] \cup [5,7) \cup \{-2\} \cup \{2\}$

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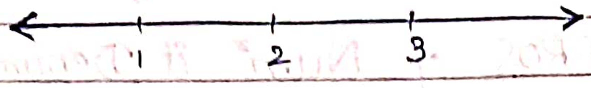
(Q) $|x-1| + |x-2| + |x-3| \leq 6.$

A) 

C-1 : $x < 1$

$|x-1| + |x-2| + |x-3| = (1-x) + (2-x) + (3-x) \leq 6$

$\Rightarrow 0 \leq x \Rightarrow 0 \leq x < 1$

C-2: $1 \leq x < 2$

$|x-1| + |x-2| + |x-3| = (x-1) + (2-x) + (3-x) \leq 6$

$\Rightarrow x \geq (-2) \Rightarrow 1 \leq x < 2$

C-3: $2 \leq x < 3$

$|x-1| + |x-2| + |x-3| = (x-1) + (x-2) + (3-x) \leq 6$

$\Rightarrow x \leq 6 \Rightarrow 2 \leq x < 3$

C-4: $3 \leq x$

$|x-1| + |x-2| + |x-3| = (x-1) + (x-2) + (x-3) \leq 6$

$\Rightarrow x \leq 4 \Rightarrow 3 \leq x \leq 4$

Ans = (C-1) $\cup$ (C-2) $\cup$ (C-3) $\cup$ (C-4)

$\Rightarrow x \in [0,1) \cup [1,2) \cup [2,3) \cup [3,4]$

$\Rightarrow \boxed{x \in [0,4]}$

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★ $\boxed{|x+y| \leq |x|+|y|}$ → ('=' when $xy \geq 0$ ⇒ Same Sign of x & y)

In general, $\boxed{|x_1+x_2+\ldots+x_n| \leq |x_1|+\ldots+|x_n|}$

★ Q) $|x-1| + |x-2| + |x-3| \leq 6$.

A) Using above prop^t, (This method does NOT work in general)

$$|3x-6| \leq |x-1|+|x-2|+|x-3| \leq 6$$

$\Rightarrow \quad |x-2| \leq 2 \Rightarrow \boxed{x \in [0,4]}$

6/04/2022

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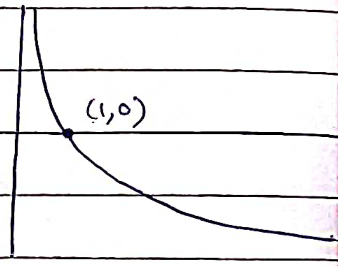
Logarithm

The logarithm of a given no. to a given base is the index of power to which base must be raised in order to equal the given no.

☆ $\boxed{\left(\log_a(N) = x\right) \Leftrightarrow \left(a^x = N\right) ; \begin{Bmatrix} a>0,\ a\neq 1, \\ N>0 \end{Bmatrix}}$

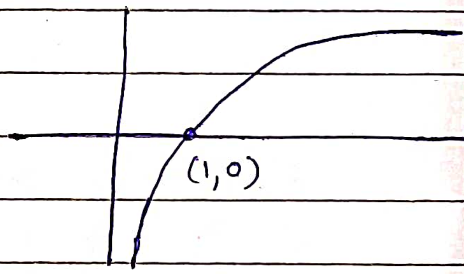
Logarithm fx^n —

$f(x) = \log_a(x)$



$a \in (0,1)$

Domain : $x \in R^+$



$a \in (1,\infty)$

Range : $x \in R$

$a \in (0,1) \Rightarrow$ (Monotonically Decreasing)

$a \in (1,\infty) \Rightarrow$ (Monotonically Increasing)

Prop's —

1) $\log_a(a) = 1$

2) $\log_a(1) = 0$

3) $a^\wedge(\log_a(x)) = x$

4) $\log_a(xy) = \log_a(x) + \log_a(y)$

5) $\log_a(x^n) = n\log_a(x)$

6) $\log_{a^n}(b^m) = \left(\frac{m}{n}\right)\log_a(b)$

7) $\log_a(b) = \frac{\log_c(b)}{\log_c(a)}$

8) $\log_a(b) = \frac{1}{\log_b(a)}$

9) $\log_a(b)\log_b(c)\log_c(d) = \log_a(d)$

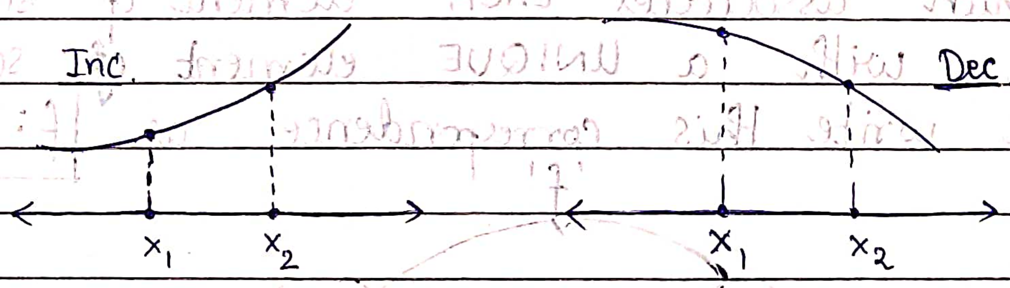
10) $a^\wedge(\log_c(b)) = b^\wedge(\log_c(a))$

08/04/2022

For any f^{x^n} 'f', if $\forall x_1, x_2 \in D_f$

$(x_1 < x_2) \Longleftrightarrow (f(x_1) < f(x_2)) \Rightarrow$ 'f' is Inc.

$(x_1 > x_2) \Longleftrightarrow (f(x_1) < f(x_2)) \Rightarrow$ 'f' is Dec.



Q) Solve $\log_{1/2}(x^2+4x+5) > 0$.

A) Domain: $(x^2+4x+5) > 0 \Rightarrow (x \in R)$

Inequality: $(x^2+4x+5) < 1 \Rightarrow (x^2+4x+4) < 0$

$\Rightarrow (x+2)^2 < 0 \Rightarrow x \in \phi$

Final Solⁿ: $(x \in R) \cap (x \in \phi) = \boxed{x \in \phi}$

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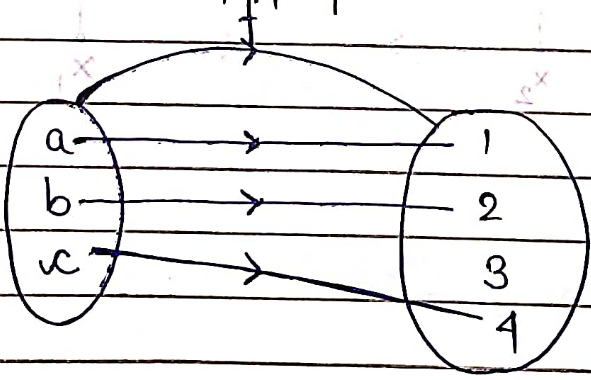
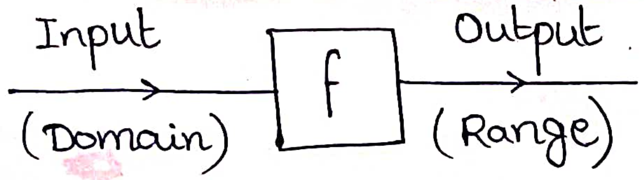
Real Fxⁿs

Function:
Consider two non empty sets A and B.
A fxⁿ "f" is a rule or correspondence which associates EACH element of set A, with a UNIQUE element of set B
We write this correspondence as $\boxed{f : A \rightarrow B}$

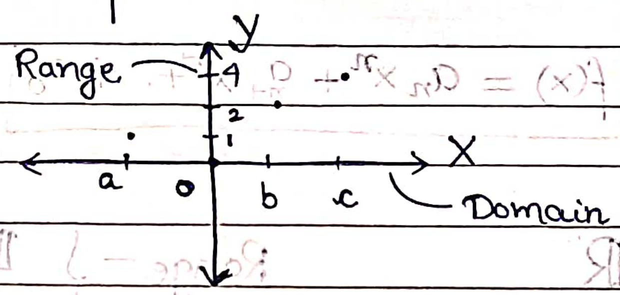


$f(a) = 1$
$f(b) = 2$
$f(c) = 4$



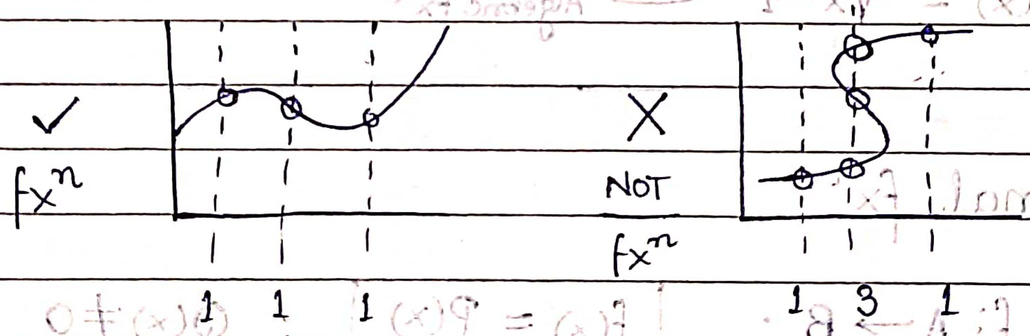
- 'a' is PRE-IMAGE of '1'
- '1' is IMAGE of 'a'.
- Set 'A' is called Domain
- Set 'B' is called Co-Domain
- Range is collection of image of elements of Set 'A' in Set 'B', under rule 'f'.

Graphical Representation —



Vertical Line Test :

If any line parallel to Y axis intersects graph at 2 or more points, the graph is NOT a fx^n.



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Domain : It is a set of pts. where fx^n is supposed to be well defined.

Eg – $f(x) = \sqrt{x}$ Domain $= [0, \infty)$

Standard $Fx^n s$ —

1) **Poly^n fx^n** :

$$\boxed{f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0} \; ; \; a_i \in \mathbb{R}$$

Domain – $\mathbb{R}$ Range – $\begin{cases} \mathbb{R} & ; n = \text{odd} \\ \text{subset of } \mathbb{R} & ; n = \text{even} \end{cases}$

2) **Algebraic fx^n** :

Eg – $f : (-\infty, -1] \cup [1, \infty) \to \mathbb{R}$

$f(x) = \sqrt{x^2 - 1}$ — Algebric fx^n.

3) **Rational fx^n** :

$f : A \to B$; $\boxed{f(x) = \dfrac{P(x)}{Q(x)}}$; $Q(x) \neq 0$

$P(x)$, $Q(x)$ are poly^n $fx^n s$.

4) Const. fx^n:

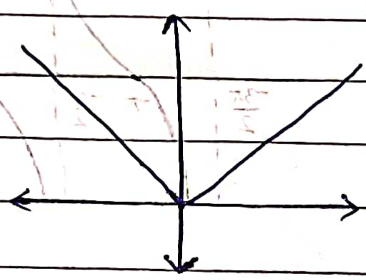
$f: \mathbb{R} \to \mathbb{R}$; $\boxed{f(x) = c}$; $c \in \mathbb{R}$

Domain — $\mathbb{R}$ Range — $\{c\}$

5) Modulus fx^n:

$f: \mathbb{R} \to \mathbb{R}$; $\boxed{f(x) = |x|}$

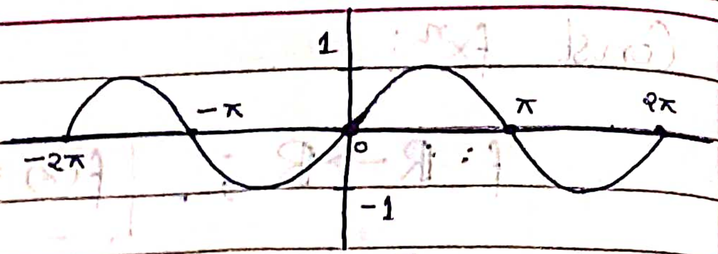
Domain — $\mathbb{R}$ Range — $[0, \infty)$



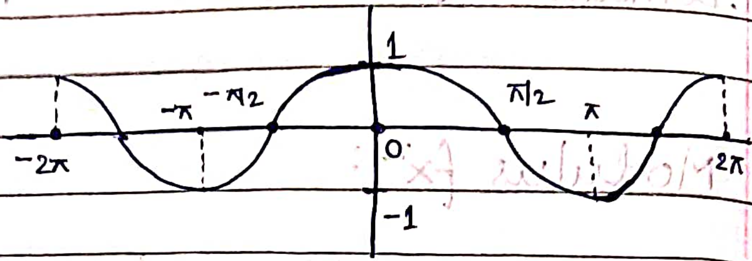
6) Trig. fx^n:

fx^n	Domain	Range
$\sin(x)$	$\mathbb{R}$	$[-1, 1]$
$\cos(x)$	$\mathbb{R}$	$[-1, 1]$
$\tan(x)$	$\mathbb{R} - \{(2n+1)\pi/2\}$	$\cancel{(-\infty, -1] \cup [1, \infty)}$ $\mathbb{R}$
$\text{cosec}(x)$	$\mathbb{R} - \{n\pi\}$	$(-\infty, -1] \cup [1, \infty)$
$\sec(x)$	$\mathbb{R} - \{(2n+1)\pi/2\}$	$\mathbb{R} - (-1, 1)$
$\cot(x)$	$\mathbb{R} - \{n\pi\}$	$\mathbb{R}$

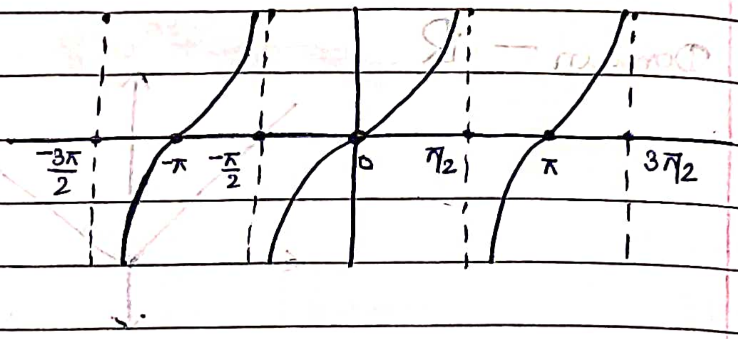
$f(x) = \sin(x)$



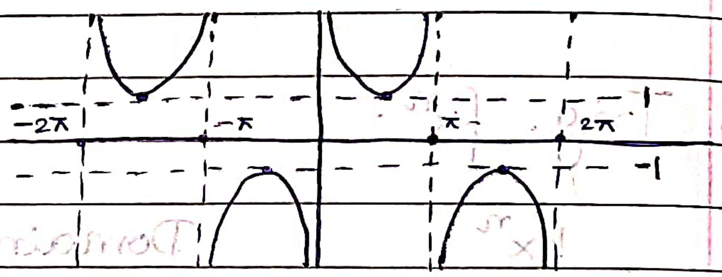
$f(x) = \cos(x)$



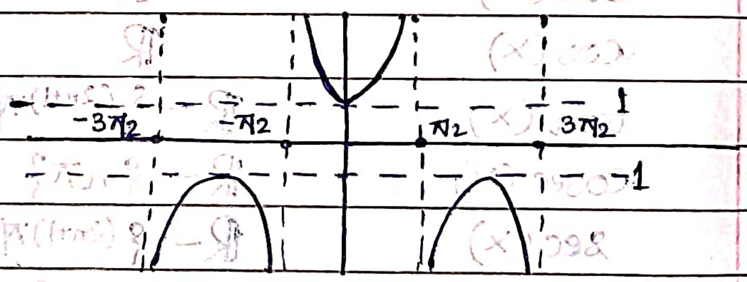
$f(x) = \tan(x)$



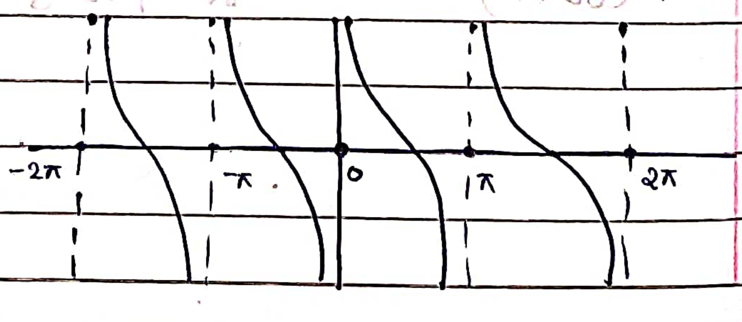
$f(x) = \text{cosec}(x)$



$f(x) = \sec(x)$



$f(x) = \cot(x)$



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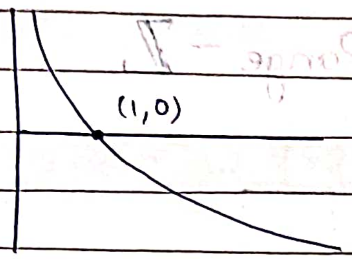
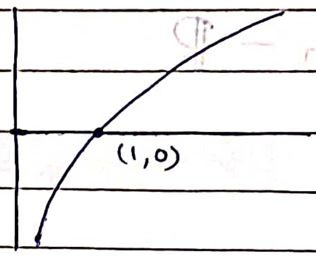
7) **Logarithmic fx^n:**

$f: \mathbb{R}^+ \to \mathbb{R}$; $\boxed{f(x) = \log_a(x)}$; $a > 0,\ a \neq 1$

Domain $- \mathbb{R}^+$ Range $- \mathbb{R}$



$0 < a < 1$



$a > 1$

Q) $\log_2(x^2 - 4x + 3) > 2$ Solve for 'x'.

A) Domain: $x^2 - 4x + 3 > 0$

Solving: $x^2 - 4x + 3 > 4 \Rightarrow x^2 - 4x - 1 > 0$

$\Rightarrow (x - (2 - \sqrt{5}))(x - (2 + \sqrt{5})) > 0$

$\Rightarrow \boxed{x \in (-\infty, 2 - \sqrt{5}) \cup (2 + \sqrt{5}, \infty)}$

☆ ~~No~~ need to solve for Domain as

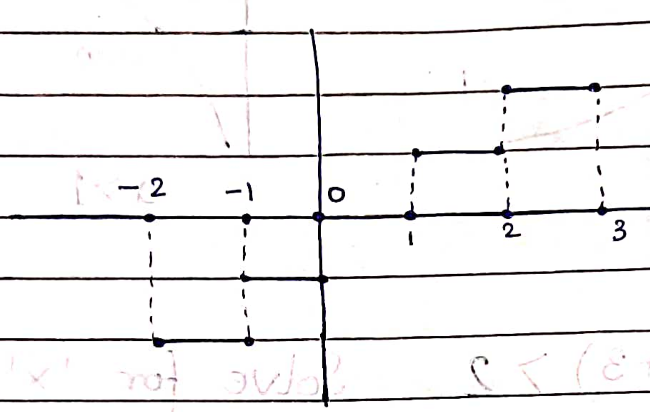
$x^2 - 4x - 1 > 0 \implies x^2 - 4x + 3 > 4 > 0$

8) Greatest Integer fx^n:

$f: \mathbb{R} \to \mathbb{R}$; $\boxed{f(x) = \lfloor x \rfloor}$

$\lfloor x \rfloor \Rightarrow$ Greatest Integer $\leq x$

Domain – $\mathbb{R}$ Range – $\mathbb{Z}$

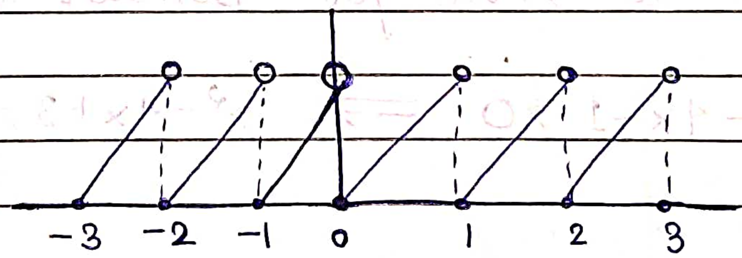


9) Fractional Part fx^n:

$f: \mathbb{R} \to \mathbb{R}$; $\boxed{f(x) = \{x\}}$

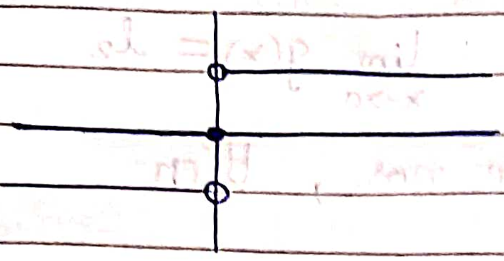
$\{x\} \Rightarrow x - \lfloor x \rfloor$

Domain – $\mathbb{R}$ Range – $[0, 1)$



10) Signum fx^n :

$$\operatorname{sgn}(x) = \begin{cases} 1 & ; \; x > 0 \\ 0 & ; \; x = 0 \\ -1 & ; \; x < 0 \end{cases}$$



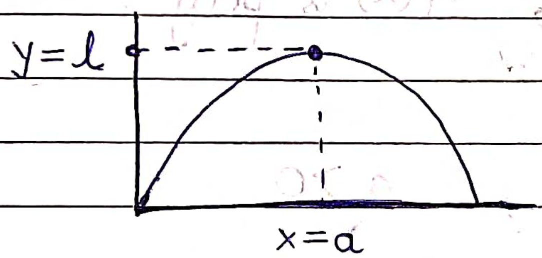
$$\operatorname{sgn}(x) = \begin{cases} |x|/x & ; \; x \neq 0 \\ 0 & ; \; x = 0 \end{cases}$$

~~Limits~~

Limits

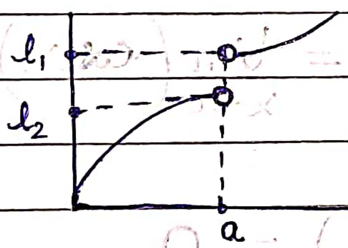
Symbol : $\boxed{\lim_{x \to a} \left(f(x) \right)}$

Limiting behaviour of fx^n $y = f(x)$ when 'x' approaches a real value 'a'.



Left Hand Limit, $\lim_{x \to a^-} \left(f(x) \right) = l$

Right Hand Limit, $\lim_{x \to a^+} \left(f(x) \right) = l$



LHL, $\lim_{x \to a^-} \left(f(x) \right) = l_2$

RHL, $\lim_{x \to a^+} \left(f(x) \right) = l_1$

☆ If 'f' exists on both sides of 'a' and LHL = RHL, then limit exists.

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Algebra of Limits —

If $\lim_{x\to a} f(x) = l_1$ and $\lim_{x\to a} g(x) = l_2$

where l_1 and l_2 are finite nos., then

1) $\lim_{x\to a}\left(c_1 f(x) \pm c_2 g(x)\right) = c_1 l_1 \pm c_2 l_2$

2) $\lim_{x\to a}\left(f(x)\cdot g(x)\right) = \left(\lim_{x\to a} f(x)\right)\left(\lim_{x\to a} g(x)\right) = l_1 l_2$

3) $\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right) = \left(\frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}\right) = \frac{l_1}{l_2}$, provided $l_2 \neq 0$

Imp. Points —

1) $\lim_{x\to a}\left(P(x)\right) = P(a)$, $P(x)$ is polyⁿ.

2) $\lim_{x\to a}\left(\frac{x^n - a^n}{x-a}\right) = na^{n-1}$, $a > 0$

3) $\lim_{x\to 0}\left(\frac{\sin(x)}{x}\right) = \lim_{x\to 0}\left(\frac{\tan(x)}{x}\right) = \lim_{x\to 0}\left(\cos(x)\right) = 1$

4) $\lim_{x\to 0}\left(\sin(x)\right) = \lim_{x\to 0}\left(\tan(x)\right) = 0$

5) If $\lim_{x\to a}(f(x))=0$, then

$$\lim_{x\to a}\left(\frac{\sin(f(x))}{f(x)}\right)=\lim_{x\to a}\left(\frac{\tan(f(x))}{f(x)}\right)=1$$

Indeterminate forms —

Form 1: 0/0 Form 2: ∞/∞

To solve these use Simplification, Factorisation, Rationalization or Trig. Substitution.

Q) $\lim_{x\to a}\left(\frac{x^{5/2}-a^{5/2}}{\sqrt{x}-\sqrt{a}}\right)$

A) $x=z^2$, $a=t^2$ $\Rightarrow$ $\lim_{z\to t}\left(\frac{z^5-t^5}{z-t}\right)=5z^4=5a^2$

Q) $\lim_{x\to 0}\left(\frac{(1+x)^5-1}{3x+5x^2}\right)$

A) $\lim_{x\to 0}\left(\frac{5x+10x^2+10x^3+5x^4+x^5}{3x+5x^2}\right)=\lim_{x\to 0}\left(\frac{5+10x+10x^2+5x^3+x^4}{3+5x}\right)$

$=5/3$

Q) $\lim_{x\to 0}\left(\frac{\sin(ax)}{\tan(bx)}\right)$ A) $\lim_{x\to 0}\left(\frac{\sin(ax)}{ax}\cdot\frac{bx}{\tan(bx)}\cdot\frac{a}{b}\right)=\frac{a}{b}$

Q) $\lim_{x\to\infty}\left(\frac{x^4+2x^3+3}{2x^4-x+2}\right)$ A) $\lim_{x\to\infty}\left(\frac{1+2/x+3/x^4}{2-1/x^3+2/x^4}\right)=\frac{1}{2}$

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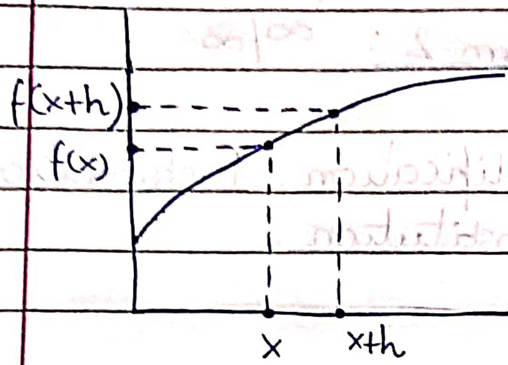
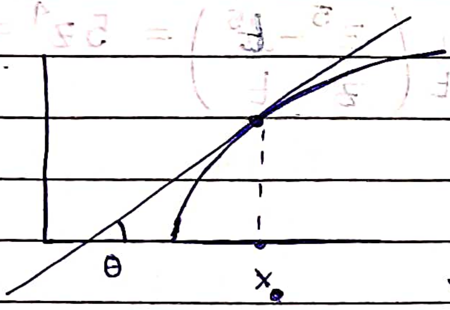
Differentiation



$\left(\frac{dy}{dx}\right)$ or $f'(x)$: differential coeff.

✓ It is rate of change of y w.r.t. change in x

$$\boxed{\left(\frac{dy}{dx}\right) = \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)} \longleftarrow \text{First Principle}$$



$$\boxed{\left(\frac{dy}{dx}\right)\Bigg|_{x=x_0} = \tan(\theta)}$$

✓ $f'(x_0)$ gives slope of tangent to $f(x)$ at $(x_0, f(x_0))$

Eg :

1) $f(x) = x^2 \longrightarrow f'(x) = \lim_{h\to 0}\left(\frac{(x+h)^2 - x^2}{h}\right) = \lim_{h\to 0}\left(\frac{x^2 + 2hx + h^2 - x^2}{h}\right)$

$\Rightarrow f'(x) = \lim_{h\to 0}\left(\frac{2hx + h^2}{h}\right) \Rightarrow \boxed{f'(x) = 2x}$

2) $f(x) = e^x \longrightarrow f'(x) = \lim_{h\to 0}\left(\frac{e^{x+h} - e^x}{h}\right) = \lim_{h\to 0}\left(e^x\left(\frac{e^h - 1}{h}\right)\right)$

$\Rightarrow \boxed{f'(x) = e^x}$

Basic Diff. formulae —

1) $y = C \Rightarrow y' = 0$

2) $y = e^x \Rightarrow y' = e^x$

3) $y = x^n \Rightarrow y' = nx^{n-1}$

4) $y = \sin(x) \Rightarrow y' = \cos(x)$

5) $y = \cos(x) \Rightarrow y' = -\sin(x)$

6) $y = \tan(x) \Rightarrow y' = \sec^2(x)$

7) $y = a^x \Rightarrow y' = a^x \ln(a)$, $a > 0$

8) $y = \ln|x| \Rightarrow y' = 1/x$

9) $y = \log_a(x) \Rightarrow y' = \dfrac{1}{x\ln(a)}$

10) $y = \sec(x) \Rightarrow y' = \sec(x)\tan(x)$

11) $y = \operatorname{cosec}(x) \Rightarrow y' = -\operatorname{cosec}(x)\cot(x)$

12) $y = \cot(x) \Rightarrow y' = -\operatorname{cosec}^2(x)$

Imp. Results

1) $y = K f(x) \Rightarrow y' = K f'(x)$

2) $y = K_1 f(x) \pm K_2 g(x) \Rightarrow y' = K_1 f'(x) \pm K_2 g'(x)$

3) (Product Rule) $y = f(x)g(x) \Rightarrow y' = f'(x)g(x) + f(x)g'(x)$

4) (Qudient Rule) $y = \dfrac{f(x)}{g(x)} \Rightarrow y' = \left(\dfrac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}\right)$

5) (Chain Rule) $y = f(g(x)) \Rightarrow y' = f'(g(x))\, g'(x)$

$$y = f(g(h(x))) \Rightarrow y' = f'(g(h(x)))\, g'(h(x))\, h'(x)$$

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8) $y = (f(x))^{g(x)} \longrightarrow \ln(y) = g(x)\ln|f(x)|$

Then differentiate.

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6) (Parametric Diff.) $x = f(t)$, $y = g(t)$

$$\left(\frac{dy}{dx}\right) = \frac{(dy/dt)}{(dx/dt)} \Rightarrow y' = \frac{g'(t)}{f'(t)}$$

7) (Implicit Diff.) Eg: $x^3y + y^2x^2 + x^3 + 9 = 0$

$$\Rightarrow 3x^2y + x^3y' + 2yy'x^2 + 2xy^2 + 3x^2 = 0$$

$$\Rightarrow y' = -\left(\frac{2xy^2 + 3x^2 + 3x^2y}{x^3 + 2yx^2}\right)$$

Higher Order Derivatives

$$y = f(x) \Rightarrow \left(\frac{dy}{dx}\right) = f'(x) \Rightarrow \left(\frac{d^2y}{dx^2}\right) = f''(x)$$

read as 'd 2 y by d x 2'.
(dee two why by dee iks two)

Indefinite Integrals

Reverse process of differentiation.

Symbol : $\int f(x)\, dx = \int y\, dx$

$f(x)$: Integrand

2nd Fundamental Theorem of Calculus –

If $\boxed{\frac{dF(x)}{dx} = f(x)} \Rightarrow \boxed{F(x) + C = \int f(x)\, dx}$

C → Const. of Integration

Imp. Results –

1) $\int K f(x)\, dx = K \int f(x)\, dx$

2) $\int (f(x) \pm g(x))\, dx = \int f(x)\, dx \pm \int g(x)\, dx$

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Basic Integration formulae –

1) $\int x^n = \left(\frac{x^{n+1}}{n+1}\right) + C, \ (n \neq -1)$

2) $\int \frac{1}{x} = \ln|x| + C$

3) $\int (f(x))^n f'(x) = \frac{(f(x))^{n+1}}{(n+1)} + C, (n \neq -1)$

4) $\int e^x = e^x + C$

5) $\int \sin(x) = -\cos(x) + C$

6) $\int \cos(x) = \sin(x) + C$

7) $\int \sec^2(x) = \tan(x) + C$

8) $\int \operatorname{cosec}^2(x) = -\cot(x) + C$

9) $\int \tan(x) = \ln|\sec(x)| + C$

10) $\int \cot(x) = \ln|\sin(x)| + C$

11) $\int \sec(x) = \ln|\sec(x) + \tan(x)| + C$

12) $\int \sec(x)\tan(x) = \sec(x) + C$

13) $\int \operatorname{cosec}(x) = \ln|\operatorname{cosec}(x) - \cot(x)| + C = \ln\left|\tan\left(\frac{x}{2}\right)\right| + C$

14) $\int \operatorname{cosec}(x)\cot(x) = -\operatorname{cosec}(x) + C$

Methods of Integration –

1) Simplification –

Eg: $\int \tan^2(x) = \int (\sec^2(x) - 1) = \tan(x) - x + C$

2) Substitution –

Eg: $\int \cos(kx)\,dx$ Let $u = kx \Rightarrow du = k\,dx$

$$= \int \frac{\cos(u)}{k}\,du = \frac{\sin(u)}{k} + C = \frac{\sin(kx)}{k} + C$$

Eg: $\int (f(x))^n f'(x)\,dx$, $(n \neq -1)$ Let $u = f(x)$ $\Rightarrow du = f'(x)\,dx$

$$= \int u^n\,du = \frac{u^{n+1}}{n+1} + C = \frac{(f(x))^{n+1}}{n+1} + C$$

Q) $\int \cos^2(x)$

A) $\int \left(\frac{1+\cos(2x)}{2}\right) = \boxed{\frac{x}{2} + \frac{\sin(2x)}{4} + C}$

Q) $\int \cos^3(x)$

A) $\int \cos(x)(1-\sin^2(x)) = \int \cos(x) - \int \sin^2(x)\cos(x)$

$$= \boxed{\sin(x) - \frac{\sin^3(x)}{3} + C}$$

Q) $\int \cos^4(x)$

A) $\int \left(\frac{1+\cos(2x)}{2}\right)^2 = \left(\frac{1}{4}\right)\left[\int 1 + 2\int c_{2x} + \int c^2_{2x}\right]$

$$= \left(\frac{1}{4}\right)\left[x + s_{2x} + \int\left(\frac{1+c_{4x}}{2}\right)\right] = \boxed{\frac{3x}{8} + \frac{s_{2x}}{4} + \frac{s_{4x}}{32} + C}$$

Q) $\int \cos^5(x)$

A) $\int (1-s^2_x)^2 c_x = \int (1-2u^2+u^4)\,du$

Let $u = s_x \Rightarrow du = c_x\,dx$ $= \boxed{s_x - \frac{2s_x^3}{3} + \frac{s_x^5}{5} + C}$

3) **By Parts** – $\boxed{\int u \, dv = uv - \int v \, du}$

Where 'u', 'v' are fx^ns of 'x'.

For choosing 'u',

u ⟶ v

I L A T E

Inverse Trig. · Log. fx^n · Algebric · Trig. fx^n · Exponential fx^n.

Eg: $\int \ln(x) = \int \ln(x) \cdot 1 \, dx$

$u = \ln(x)$

$dv = dx$

$\Rightarrow v = x$

$= x\ln(x) - \int x \cdot \frac{1}{x} \, dx$

$= \boxed{x\ln(x) - x + C}$

Q) Let $I_n = \int (\ln(x))^n \, dx$. Find its recurrence rel^n.

A) Let $u = (\ln(x))^n$, $v = x$

$\int (\ln(x))^n \cdot 1 \, dx = x(\ln(x))^n - \int x \cdot \frac{n(\ln(x))^{n-1}}{x} \, dx$

$\Rightarrow \boxed{I_n = x(\ln(x))^n - n I_{n-1}}$

Definite Integrals

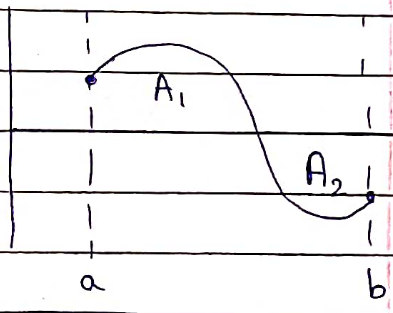
Symbol: $\int_a^b f(x)\,dx$; a – Lower Limit
b – Upper Limit

Fundamental Theorem of Calculus –

If $\boxed{\int f(x)\,dx = F(x) + C} \Rightarrow \boxed{\int_a^b f(x)\,dx = F(b) - F(a)}$

Geometrical Meaning :

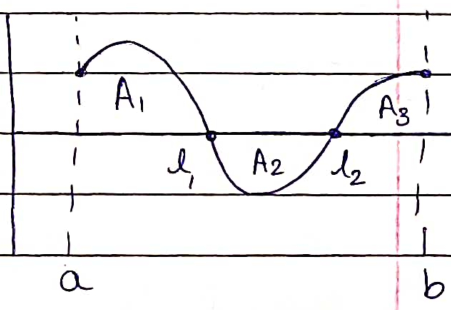
$$\boxed{\int_a^b f(x)\,dx = A_1 - A_2}$$



Area Under Curve –

$$\boxed{\text{Area} = \int_a^b |f(x)|\,dx}$$

$\cancel{A_1 + A_2 + A_3}$



$$= \left|\int_a^{l_1} f(x)\,dx\right| + \left|\int_{l_1}^{l_2} f(x)\,dx\right| + \left|\int_{l_2}^{b} f(x)\,dx\right|$$

Area b/w $y = f(x)$, $x = a$, $x = b$, $y = 0$.

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Evaluation of Definite Integrals —

1) Substitution :

$$\int_0^{\frac{\pi}{2}} \sin^2(x)\cos(x)\,dx$$

Let $t = \sin(x)$
$\Rightarrow dt = \cos(x)$

x	0	$\pi/2$
t	0	1

$$= \int_0^1 t^2\,dt = \left(\frac{t^3}{3}\right)_0^1$$

$$= 1/3$$

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Trigonometry

Imp. Result —

☆ $$\boxed{-\sqrt{a^2+b^2} \leq a\sin(\theta)+b\cos(\theta) \leq \sqrt{a^2+b^2}}$$

Proof : Let $f(\theta) = a\sin(\theta) + b\cos(\theta)$

$$= \left(\sqrt{a^2+b^2}\right)\left(\frac{a}{\sqrt{a^2+b^2}}\sin(\theta) + \frac{b}{\sqrt{a^2+b^2}}\cos(\theta)\right)$$

Since $\left(\frac{a}{\sqrt{a^2+b^2}}\right) \in [-1, 1]$; let it $= \sin(\alpha)$

$$\Rightarrow f(\theta) = \left(\sqrt{a^2+b^2}\right)\left(\sin(\alpha)\sin(\theta) + \cos(\alpha)\cos(\theta)\right)$$

$$\Rightarrow f(\theta) = \left(\sqrt{a^2+b^2}\right)\cos(\theta-\alpha)$$

$$\Rightarrow \left(-\sqrt{a^2+b^2}\right) \leq f(\theta) \leq \left(\sqrt{a^2+b^2}\right)$$

Allied Angles —

$$\boxed{\begin{aligned} T(\pi n \pm \theta) &= \circ\, T(\theta) \\ T\left(\frac{\pi}{2}(2n+1) \pm \theta\right) &= \circ\, T^c(\theta) \end{aligned}}$$

Sign using quadrant & ASTC Rule

$T \rightarrow$ Trig. , $T^c \rightarrow$ Trig. Complement.

Compound Angle —

$$\sin(A+B)\sin(A-B) = \sin^2(A) - \sin^2(B) = \cos^2(B) - \sin^2(A)$$

$$\cos(A+B)\cos(A-B) = \cos^2(A) - \sin^2(B) = \cos^2(B) - \sin^2(A)$$

$$\tan(\theta_1+\theta_2+\theta_3) = \left(\frac{\sum t_{\theta_1} - \prod t_{\theta_1}}{1 - \sum t_{\theta_1} t_{\theta_2}}\right)$$

$$\tan(\theta_1 + \ldots + \theta_n) = \left(\frac{S_1 - S_3 + S_5 \ldots}{1 - S_2 + S_4 \ldots}\right)$$

where $S_2 = \sum t_{\theta_1} t_{\theta_2}$, $S_3 = \sum t_{\theta_1} t_{\theta_2} t_{\theta_3}$, ...

☆Q) Prove that $\cos(\sin(\theta)) > \sin(\cos(\theta))$ $\forall \theta \in [0, \pi/2]$.

A) $\cos(\sin(\theta)) > \sin(\cos(\theta))$

$\Rightarrow \sin\left(\pi/2 - \sin(\theta)\right) > \sin(\cos(\theta))$

$\Rightarrow \pi/2 - \sin(\theta) > \cos(\theta) \Rightarrow \sin(\theta) + \cos(\theta) < \pi/2$

Since all steps reversible, hence proven.

☆Q) Find max. value of $\left(\begin{array}{c} 4\sin^2(x) + 3\cos^2(x) \\ + \sin(x/2) + \cos(x/2) \end{array}\right)$

A) Observe that $4\sin^2(x) + 3\cos^2(x) = 3 + \sin^2(x) \leqslant 4$

and $\sin(x/2) + \cos(x/2) \leqslant \sqrt{2}$

Since both max. values occur at $x = \pi/2$

$\boxed{\text{Max. value} = 4 + \sqrt{2}}$

Multiple Angle —

$\sin(3A) = 4\sin(A + 60^\circ)\sin(A)\sin(60^\circ - A)$

$\cos(3A) = 4\cos(60^\circ + A)\cos(A)\cos(60^\circ - A)$

$\tan(3A) = \tan(60^\circ + A)\tan(A)\tan(60^\circ - A)$

Standard Values —

$\sin(15^\circ) = \left(\dfrac{\sqrt{3}-1}{2\sqrt{2}}\right)$; $\cos(15^\circ) = \left(\dfrac{\sqrt{3}+1}{2\sqrt{2}}\right)$

$\tan(15^\circ) = (2 - \sqrt{3})$; $\tan(22.5^\circ) = (\sqrt{2} - 1)$

$\sin(22.5^\circ) = \left(\dfrac{\sqrt{2-\sqrt{2}}}{2}\right)$; $\cos(22.5^\circ) = \left(\dfrac{\sqrt{2+\sqrt{2}}}{2}\right)$

$\sin(18^\circ) = \left(\dfrac{\sqrt{5}-1}{4}\right)$; $\cos(36^\circ) = \left(\dfrac{\sqrt{5}+1}{4}\right)$

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Imp. Series –

$$\sin(A) + \sin(A+2B) + \ldots + \sin(A+(n-1)B) = \left(\frac{\sin\left(A+(n-1)\frac{B}{2}\right)\sin\left(\frac{nB}{2}\right)}{\sin(B/2)}\right)$$

$$\cos(A) + \cos(A+2B) + \ldots + \cos(A+(n-1)B) = \left(\frac{\cos\left(A+(n-1)B/2\right)\sin(nB/2)}{\sin(B/2)}\right)$$

$$\cos(A)\cos(2A)\ldots\cos(2^{n-1}A) = \left(\frac{\sin(2^n A)}{2^n \sin(A)}\right)$$

☆ $$\boxed{\sum_{k=0}^{n-1}\left(\cos\left(\frac{2k\pi}{n}\right)\right) = \sum_{k=0}^{n-1}\left(\sin\left(\frac{2k\pi}{n}\right)\right) = 0}$$

Conditional Identities –

If $A+B+C=\pi$, then

$\sum s_{2A} = 4\prod s_A$ $\qquad$ $\sum c_{2A} = -1-4\prod c_A$

$\sum s_A = 4\prod c_{A/2}$ $\qquad$ $\sum c_A = 1+4\prod s_{A/2}$

$\sum t_A = \prod t_A$ $\qquad$ $\sum t_{A/2}t_{B/2} = 1$

Imp. Inequalities -

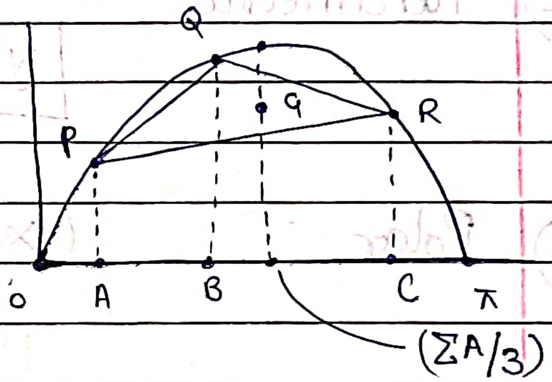
If $A+B+C=\pi$, then

$\sum s_A \leq \left(\frac{3\sqrt{3}}{2}\right)$ $\qquad$ $\sum c_A \leq \left(\frac{3}{2}\right)$

$\sum t^2_{A/2} \geq 1$ $\qquad$ $\sum t_A \geq 3\sqrt{3}$ (if all angles acute)

☆ Q) Prove that $\sum s_A \leq \left(\frac{3\sqrt{3}}{2}\right)$, if $A+B+C=\pi$.

A) $P \equiv (A, s_A)$
$Q \equiv (B, s_B)$
$R \equiv (C, s_C)$



$\Rightarrow G \equiv \left(\frac{\sum A}{3}, \frac{\sum s_A}{3}\right)$

Now, G below curve $\Rightarrow \left(\frac{\sum s_A}{3}\right) \leq s_{\left(\frac{\sum A}{3}\right)}$

$\Rightarrow \boxed{\sum s_A \leq \left(\frac{3\sqrt{3}}{2}\right)}$ (equality when $A=B=C=\pi/3$)